Generalized local interactions in 1D:

Solutions of quantum many-body systems describing distinguishable particles

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GK-Seminar “Quantenfeldtheorie und Mathematische Physik”

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Introduction

- Low energy properties of an interacting quantum system: replace details of interaction by local singular interaction at one point
- Most prominent example: δ-interaction $\longrightarrow$ boson gas [Lieb, Liniger ’63]
- Selfadjoint extensions of free Hamiltonian in $1D$ parametrized by four real parameters
- Q: Which models allow a solvable many-body generalization?
- Why $1D$?
  - Mathematically tractable $\longrightarrow$ testing ground for numerical methods
  - Recently experimental realization of $1D$ quantum systems
  - Beautiful mathematics: Yang-Baxter equation [Yang ’67]

Plan of this talk

- Selfadjoint extensions of the free Hamiltonian in $1D$
- Many-body generalization
- Coordinate Bethe ansatz
- Yang-Baxter equation and Zamolodchikov algebra
- Complete set of solutions of the YBE
- Discussion

Selfadjoint extensions of the free Hamiltonian

- Consider free Hamiltonian in $1D$

$$\int_{|x|>0} dx \left(\overline{\varphi''(x)}\,\psi(x) - \overline{\varphi(x)}\,\psi''(x)\right) = 0 \quad \Rightarrow \quad [\overline{\varphi'}\,\psi - \overline{\varphi}\,\psi']_{x=0^+} = [\overline{\varphi'}\,\psi - \overline{\varphi}\,\psi']_{x=0^-}$$

- Known: s.a. extensions define four real parameter family

$$V(x) = c\,\delta(x) + 4\lambda\,\partial_x\,\delta(x)\,\partial_x + 2(\gamma + i\eta)\partial_x\,\delta(x) - 2(\gamma - i\eta)\,\delta(x)\,\partial_x$$

- Regularize jumps at $x = 0$: $\psi(0) \mapsto \frac{\psi(0^+)+\psi(0^-)}{2}$, $\psi'(0) \mapsto \frac{\psi'(0^+)+\psi'(0^-)}{2}$

- Integrate eigenvalue eq. twice $\longrightarrow$ boundary condition

$$\psi'(0^+) - \tfrac{c}{2}\psi(0^+) + (\gamma - i\eta)\psi'(0^+) = \psi'(0^-) + \tfrac{c}{2}\psi(0^-) - (\gamma - i\eta)\psi'(0^-)$$
$$\psi(0^+) - 2\lambda\psi'(0^+) - (\gamma + i\eta)\psi(0^+) = \psi(0^-) + 2\lambda\psi'(0^-) + (\gamma + i\eta)\psi(0^-)$$

Many-body generalization

- Only two-body interactions:

$$H = -\sum_j \partial_{x_j}^2 + \sum_{j<k} V_{jk}$$

$$V_{jk} = 2c\delta(x_j - x_k) + 2\lambda(\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k})$$
$$+2(\gamma + i\eta)(\partial_{x_j} - \partial_{x_k})\delta(x_j - x_k) - 2(\gamma - i\eta)\delta(x_j - x_k)(\partial_{x_j} - \partial_{x_k})$$

- Boundary conditions: pick j, k, $r = x_j - x_k$, $s = x_j + x_k$, integrate twice w.r.t. r variable ($\psi_\pm = \psi|_{x_j = x_k + 0^\pm}$)

$$(\partial_{x_j} - \partial_{x_k})\left[\psi_+ - \psi_-\right] = c\left[\psi_+ + \psi_-\right] - (\gamma - i\eta)\left(\partial_{x_j} - \partial_{x_k}\right)\left[\psi_+ + \psi_-\right]$$
$$\psi_+ - \psi_- = \lambda(\partial_{x_j} - \partial_{x_k})\left[\psi_+ + \psi_-\right] + (\gamma + i\eta)\left[\psi_+ + \psi_-\right]$$

Solution of the two-body case

- Superposition of plane waves

$$\varphi(x_1,x_2) = \begin{cases} e^{\mathrm{i}(k_1x_1+k_2x_2)} + S_R^+(k_1-k_2)e^{\mathrm{i}(k_2x_1+k_1x_2)} & x_1 < x_2 \\ S_T^+(k_1-k_2)e^{\mathrm{i}(k_1x_1+k_2x_2)} & x_1 > x_2 \end{cases}$$

- Substitution into b.c. gives $S^+_{R,T}$ as functions of c,λ,γ,η

- Symmetry $x_1 \leftrightarrow x_2$, $(\gamma,\eta) \mapsto (-\gamma,-\eta)$ of H gives linearly independent solution $\tilde{\varphi}$ with

$$S^-_{R,T} = S^+_{R,T}\big|_{(\gamma,\eta)\mapsto(-\gamma,-\eta)}$$

- Complete set of eigenfunctions

The coordinate Bethe ansatz

- Consider wedge Δ_Q with $x_{Q(1)} < \ldots < x_{Q(N)}$ for $Q \in S_N$
- Inside wedges: plane waves $\psi(x) = \sum_{P\in S_N} A_P(Q)\, e^{\mathrm{i}k_P x_Q}$ for $x \in \Delta_Q$.
- Choose Q, $0 < i \leq N$ s.t. $Q(i) = j$, $Q(i+1) = k$, $j < k$. The b.c. for x_j, x_k reads

$$\begin{aligned}
\mathrm{i}(k_{P(i)} - k_{P(i+1)})[A_{PT_i}(QT_i) - A_P(QT_i) - A_P(Q) + A_{PT_i}(Q)] \\
= c[A_P(Q) + A_{PT_i}(Q) + A_P(QT_i) + A_{PT_i}(QT_i)] \\
- (\mathrm{i}\gamma + \eta)[A_{PT_i}(QT_i) - A_P(QT_i) + A_P(Q) - A_{PT_i}(Q)]
\end{aligned}$$

$$\begin{aligned}
A_P(QT_i) + A_{PT_i}(QT_i) - A_P(Q) - A_{PT_i}(Q) \\
= \mathrm{i}\lambda(k_{P(i)} - k_{P(i+1)})[A_{PT_i}(QT_i) - A_P(QT_i) + A_P(Q) - A_{PT_i}(Q)] \\
+ (\gamma + \mathrm{i}\eta)[A_P(Q) + A_{PT_i}(Q) + A_P(QT_i) + A_{PT_i}(QT_i)]
\end{aligned}$$

- Solutions? Problem: $(N!)^2$ unknowns $A_P(Q)$, $(N-1) \times (N!)^2$ equations $\Rightarrow$ overdetermined!
- Consistent solutions: $N!$ parameters $A_P(I)$

The coordinate Bethe ansatz II

- If the Bethe ansatz is consistent, then

$$A_{PT_i}(Q) = S_R^+(k_{P(i)} - k_{P(i+1)}) A_P(Q) + S_T^-(k_{P(i)} - k_{P(i+1)}) A_P(QT_i)$$
$$A_{PT_i}(QT_i) = S_R^-(k_{P(i)} - k_{P(i+1)}) A_P(QT_i) + S_T^+(k_{P(i)} - k_{P(i+1)}) A_P(Q)$$

- It is instructive to look at two-body case: General solution of b.c. is

$$\begin{aligned} \psi &= a_1 \varphi + a_2 \tilde{\varphi} \\ &= \begin{cases} a_1 e^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + f_1(k_1 - k_2) e^{\mathrm{i}(k_2 x_1 + k_1 x_2)}, & x_1 < x_2 \\ f_2(k_1 - k_2) e^{\mathrm{i}(k_2 x_2 + k_1 x_1)} + a_2 e^{\mathrm{i}(k_1 x_2 + k_2 x_1)}, & x_2 < x_1 \end{cases} \end{aligned}$$

where

$$\begin{aligned} f_1(\cdot) &= S_R^+(\cdot) a_1 + S_T^-(\cdot) a_2 \\ f_2(\cdot) &= S_R^-(\cdot) a_2 + S_T^+(\cdot) a_1 \end{aligned}$$

- Note that $x_2 < x_1 \Leftrightarrow x_{T_1(1)} < x_{T_1(2)} \Leftrightarrow x \in \Delta_{T_1}$. Thus,

$$a_1 = A_I(I), \quad a_2 = A_I(T_1), \quad f_1 = A_{T_1}(I), \quad f_2 = A_{T_1}(T_1).$$

- General case: $x_1 \mapsto x_{Q(i)}$, $k_1 \mapsto k_{P(i)}$ etc.

Consistency of the ansatz: Yang-Baxter equation

- Organize $A_P(Q)$ in columns A_P and write $(\mathbb{T}_i A_P)(Q) = A_P(QT_i)$, then

$$A_{PT_i} = \mathbb{Y}_i(k_{P(i)} - k_{P(i+1)})A_P$$

  for $\mathbb{Y}_i(\cdot) = \mathbb{S}^i_R(\cdot) + \mathbb{S}^i_T(\cdot)\mathbb{T}_i$, where $\mathbb{S}^i_{R,T}$ are diagonal with entries $S^{\pm}_{R,T}$

- Can succesively compute A_P from A_I by decomposing $P = T_{i_1} \cdots T_{i_r}$

- Consistency from relations in S_N

$$T_i T_i = 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad |i-j| > 1$$

- (Generalized) Yang-Baxter equations:

$$\mathbb{Y}_i(-u)\mathbb{Y}_i(u) = \mathbb{1}, \quad \mathbb{Y}_i(v)\mathbb{Y}_{i+1}(u+v)\mathbb{Y}_i(u) = \mathbb{Y}_{i+1}(u)\mathbb{Y}_i(u+v)\mathbb{Y}_{i+1}(v)$$

Solving the YBE

- Sufficient to check system of three particles

- Two strategies

1. Construct an ordering of S_N that respects $S_N \hookrightarrow S_{N+1}$ and gives a decomposition of each element into transpositions
   Calculate $\mathbb{S}^i_{R,T}$ for three particles
   Solve system of equations for c, λ, γ, η

2. Use Zamolodchikov algebra to deduce this system of equations

Zamolodchikov algebra

- Assign to every $A_P(Q)$ a permutation of $A(k_1)\,B(k_2)\,\ldots$
- $A(k_i),\ldots$ generate associative algebra with relation

$$A(k_1)\,B(k_2) = S_R^{AB}(k_1-k_2)\,A(k_2)\,B(k_1) + S_T^{AB}(k_1-k_2)\,B(k_2)\,A(k_1)$$

- Set $S_{R,T}^{AB}(\cdot) = S_{R,T}^{\pm}(\cdot)$
- Consistency relations

$$\begin{aligned}
A(k_1)\,B(k_2) &= S_R^{AB}(k_1-k_2)\left[S_R^{AB}(k_2-k_1)\,A(k_1)\,B(k_2) + S_T^{AB}(k_2-k_1)\,B(k_1)\,A(k_2)\right] \\
&\quad + S_T^{AB}(k_1-k_2)\left[S_R^{BA}(k_2-k_1)\,B(k_1)\,A(k_2) + S_T^{BA}(k_2-k_1)\,A(k_1)\,B(k_2)\right] \\
&= \left[S_R^{AB}(k_1-k_2)\,S_R^{AB}(k_2-k_1) + S_T^{AB}(k_1-k_2)\,S_T^{BA}(k_2-k_1)\right]A(k_1)\,B(k_2) \\
&\quad + \left[S_R^{AB}(k_1-k_2)\,S_T^{AB}(k_2-k_1) + S_T^{AB}(k_1-k_2)\,S_R^{BA}(k_2-k_1)\right]B(k_1)\,A(k_2)
\end{aligned}$$

Solutions of the YBE

- 4 quadratic, 9 cubic equations for the S's:

$$S_R^+(v)\,S_R^+(u+v)\,S_T^-(u)+S_T^-(v)\,S_R^+(u+v)\,S_R^-(u)=S_R^+(u)\,S_T^-(u+v)\,S_R^+(v)$$

- Substitute expression for $S_{R,T}^{\pm}$ yields solutions

$$\gamma=\lambda=0 \qquad \text{or} \qquad \lambda=1/c,\gamma=\eta=0$$

Discussion I

- Solution of type $(c, 1/c, 0, 0)$:
  - Two-body Hamiltonian commutes with transposition
  - Choice of exchange statistics (Yang tableau) determines wave function on all wedges once it is known in Δ_I
  - $S_R^+ = S_R^-, S_T^+ = S_T^- = 0 \Rightarrow \mathbb{Y}_i = Y_i \mathbb{1}$, where $Y_i(u) = \frac{\mathrm{i}u+c}{\mathrm{i}u-c}$
  - Y_i identical to δ-interaction on bosons: Can write down wave functions

$$\psi(x) \sim \prod_{j<k} (\partial_{x_j} - \partial_{x_k} + c) \det_{m<n} [\exp(\mathrm{i}k_m x_n)] \text{ for } x \in \Delta_I$$

  - Bosons: $\partial\delta\partial$ interaction invisible, non-relativistic limit of “sine-Gordon” model
  - Fermions: δ interaction invisible, non-relativistic limit of massive Thirring model
  - $\lambda = 1/c$ non-relativistic remnant of Coleman duality (?)

Discussion II

- Solution of type $(c,0,0,\eta)$:
  - Observation: If ψ obeys b.c. for $(c,0,0,\eta)$ then

$$\tilde{\psi}(x) = e^{-\mathrm{i}\alpha\theta(x)}\psi(x), \quad e^{\mathrm{i}\alpha} = \frac{1+\mathrm{i}\eta}{1-\mathrm{i}\eta}$$

    satisfies b.c. for pure δ case $(\frac{c}{1+\eta^2},0,0,0)$
  - Moreover

$$S_T^{\pm}(u) = e^{\pm\mathrm{i}\alpha}b(u), \quad S_R^{\pm}(u) = a(u),$$

    where $Y_i(u) = a(u) + b(u)\mathbb{T}_i$ defines rational solution of the YBE
  - Solution unitarily equivalent to the δ interaction case

Summary

- Determined all quantum solvable models in $1D$ which come from four parameter family of s.a. extensions of free Hamiltonian
- Analysis not restricted to distinguishable particles
- Generalization of Yang-Baxter equations
- Found two families of solutions:
  1. one parameter model for which eigenfunctions can be given explicitly
  2. one parameter extension of δ interaction to distinguishable particles which is unitarily equivalent to δ interaction
- Other local two-body interactions? $i\delta(x_j - x_k)(\partial_{x_j} + \partial_{x_k})$, "derivative nonlinear Schrödinger equation", is solvable by coordinate Bethe ansatz.